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Effect of viscous dissipation on the asymptotic behaviour of laminar forced convection in circular tubes

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Abstract—The asymptotic behaviour of laminar forced convection in a circular tube, for a Newtonian fluid at constant properties, is analysed by taking into account the viscous dissipation effects. The axial heat conduction in the fluid is neglected. A sufficient condition for the existence of a fully developed region is determined. This condition includes, for instance, any asymptotically vanishing axial distribution of the wall heat flux, uniform wall temperature, convection with an external fluid. The asymptotic temperature field and the asymptotic value of the Nusselt number are determined analytically, for every boundary condition which allows a fully developed region. In particular, it is proved that, whenever the wall heat flux tends to zero, the asymptotic Nusselt number is zero. Copyright © 1996 Elsevier Science Ltd.

INTRODUCTION

In a recent paper [1], the asymptotic behaviour of laminar forced convection in circular ducts has been analysed under the assumptions that both the viscous dissipation and the axial conduction in the fluid can be neglected. In particular, a sufficient condition much broader than those previously found in the literature has been determined for the existence of a fully developed region. In this paper, the asymptotic behaviour of laminar forced convection in circular tubes is analysed by taking into account the effects of viscous dissipation. A very broad sufficient condition for the existence of an asymptotic thermally developed region is determined. The asymptotic temperature field and the asymptotic value of the Nusselt number are evaluated analytically for every boundary condition which yields a fully developed region. The results show that in many cases, such as whenever the wall heat flux tends to zero, the effect of viscous dissipation on the fully developed laminar forced convection is very important.

In the literature on laminar forced convection in circular tubes, the effect of viscous dissipation is almost always neglected. Indeed, this effect is usually considered as relevant, in convection heat transfer, only in two cases: flow in capillary tubes; flow of very viscous fluids. One of the earliest papers in which the effect of viscous dissipation is studied concerns the forced convection in capillary tubes employed for measurements of viscosity [2]. In this reference, two boundary conditions are considered: zero wall heat flux and uniform wall temperature. For these boundary conditions, the temperature distribution in the thermal entrance region is determined. More recently, the viscous dissipation effect has been analysed under the boundary conditions of uniform wall temperature [3] and of uniform wall heat flux [3, 4]. A wide study on the effect of viscous dissipation on the fully developed laminar forced convection in cylindrical tubes with arbitrary cross-section and uniform wall temperature has been performed in [5]. A very interesting result obtained in the literature, on the laminar forced convection with viscous dissipation in circular tubes, is as follows. If the boundary condition is either uniform wall temperature or convection, with a uniform convection coefficient, to an external fluid having a uniform temperature outside its boundary layer, then the value of the fully developed Nusselt number is $48/5 = 9.6$, independently of the thermophysical properties or the mean velocity of the fluid and of the tube radius [3, 5, 6, 7]. This result points out that the effects of viscous dissipation on the fully developed laminar forced convection in circular tubes may be important for every value of the fluid viscosity. In particular, for the boundary conditions of uniform wall temperature and of convection to an isothermal external fluid, the hypothesis of negligible viscous dissipation effects leads to completely misleading results. For instance, in the case of uniform wall temperature, if viscous dissipation effects are neglected the well known value of the asymptotic Nusselt number 3.6568 is obtained, which is very far from the correct value 9.6. The aim of the present paper is to study the asymptotic behaviour, i.e. far from the inlet section, of the temperature field in a fluid with constant density, in a steady and laminar motion in a circular tube, by taking into account viscous dissipation effects. Any axial distribution of the wall heat flux is considered. In particular, the analysis points out in

NOMENCLATURE

A	constant [K]
a_0	dimensionless constant defined by equation (48)
Bi	Biot number, $h_e r_0/k$
Br	Brinkman number, $\mu \bar{u}^2/(2r_0 q_w)$
C	constant [K]
c_n	dimensionless coefficient defined by equation (47)
$\mathscr{F}$	function defined by equation (9) [$m^3\ s^{-1}\ K^2$]
F	dimensionless function employed in equation (23)
f	limit of F for $x \to +\infty$
h_e	external convection coefficient [$W\ m^{-2}\ K^{-1}$]
k	thermal conductivity [$W\ m^{-1}\ K^{-1}$]
n	natural number
Nu	Nusselt number, $2r_0 q_w/[k(T_w - T_b)]$
Pe	Peclet number, $2\bar{u}r_0/\alpha$
q_w	wall heat flux [$W\ m^{-2}$]
r	radial coordinate [m]
r_0	tube radius [m]
$\Re$	set of the real numbers
s	dimensionless radial coordinate, r/r_0
T	temperature [K]
T_1, T_2, $\tilde{T}$	temperature fields [K]
T_0, $T_0^{(1)}$, $T_0^{(2)}$	inlet temperature profiles [K]
T_e	temperature of the external fluid [K]
u	velocity component in the axial direction [$m\ s^{-1}$]
$\bar{u}$	mean value of u [$m\ s^{-1}$]
x	axial coordinate [m]
x'	integration variable [m]
X	any function of r and x.

Greek symbols

α	thermal diffusivity [$m^2\ s^{-1}$]
β	constant defined by equation (39) [m^{-1}]
$\hat{\beta}$	dimensionless constant, $Pe\, r_0 \beta$
ϑ	dimensionless temperature, $(T_w - T)/(T_w - T_b)$
μ	dynamic viscosity [Pa s]
Φ	viscous dissipation function [s^{-2}].

Subscripts

b	bulk quantity
w	quantity evaluated at the wall
∞	limit of a quantity for $x \to +\infty$.

which cases the effect of viscous dissipation on the asymptotic dimensionless temperature field cannot be neglected.

ANALYSIS OF THE BOUNDARY VALUE PROBLEM

In this section, the boundary value problem which describes the laminar and forced convection in a circular tube, with a prescribed axial distribution of wall heat flux, is analysed under the assumption that the axial heat conduction in the fluid is negligible. In particular, the uniqueness of the solution of the boundary value problem is proved, and the dependence of the asymptotic temperature field on the inlet temperature distribution is studied.

Reference will be made to the laminar and forced convection in a circular duct with radius r_0 for a newtonian fluid with a constant thermal diffusivity α, a constant thermal conductivity k and a constant dynamic viscosity μ. The velocity field in the fluid will be supposed completely developed and the axial heat conduction in the fluid will be considered as negligible. If the axial distribution of the wall heat flux $q_w(x)$ and the temperature distribution in the inlet section $T_0(r)$ are prescribed, the temperature field $T(r, x)$ is determined by the energy equation [8]

$$\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = \frac{ru(r)}{\alpha}\frac{\partial T}{\partial x} - \frac{\mu}{k}r\Phi(r) \qquad (1)$$

and by the boundary conditions

$$k\left.\frac{\partial T}{\partial r}\right|_{r=r_0} = q_w(x) \qquad (2)$$

$$T(r, 0) = T_0(r) \qquad (3)$$

where $0 \leqslant r \leqslant r_0$ and $x \geqslant 0$. The function $u(r)$ is Hagen–Poiseuille's velocity distribution

$$u(r) = 2\bar{u}\left(1 - \frac{r^2}{r_0^2}\right) \qquad (4)$$

while $\Phi(r)$ is the viscous dissipation function [8] which, by employing equation (4), can be expressed in the form

$$\Phi(r) = \left[\frac{\mathrm{d}u(r)}{\mathrm{d}r}\right]^2 = \frac{16\bar{u}^2}{r_0^4}r^2. \qquad (5)$$

In addition to the boundary conditions (2) and (3), one has $\partial T/\partial r|_{r=0} = 0$ for every $x \geqslant 0$, on account of cylindrical symmetry and of the regularity of $T(r, x)$.

In the following, the uniqueness of the solution of the energy equation (1), under the boundary conditions (2) and (3), will be proved. Namely, it will be proved that if $T_1(r,x)$ and $T_2(r,x)$ are two temperature fields which obey equations (1)–(3), then $T_1(r,x) = T_2(r,x)$ for every r and for every x. Indeed, on account of equations (1)–(3), the function $\tilde{T}(r,x) = T_1(r,x) - T_2(r,x)$ obeys the equation

$$\frac{\partial}{\partial r}\left(r\frac{\partial \tilde{T}}{\partial r}\right) = \frac{ru(r)}{\alpha}\frac{\partial \tilde{T}}{\partial x} \tag{6}$$

with the boundary conditions:

$$\left.\frac{\partial \tilde{T}}{\partial r}\right|_{r=r_0} = 0 \tag{7}$$

$$\tilde{T}(r,0) = 0. \tag{8}$$

It has been already proved by Barletta and Zanchini [1] that the boundary value problem expressed by equations (6)–(8) has the unique solution $\tilde{T}(r,x) = 0$ for every r and for every x. Therefore, the solution of the boundary value problem given by equations (1)–(3) is unique. Along the same lines, it can be easily proved that the solution of equation (1) is unique even if the boundary condition (2) is replaced by a boundary condition of the first kind, i.e. if the temperature distribution at the wall is prescribed.

Let us now determine how the asymptotic behaviour of the temperature field which obeys conditions (1)–(3) depends on the boundary condition (3), i.e. on the temperature distribution in the inlet section. Let $T_1(r,x)$ be a temperature field which obeys equations (1) and (2) and the boundary condition in $x = 0$

$$T(r,0) = T_0^{(1)}(r) \tag{9}$$

and let $T_2(r,x)$ be a temperature field which obeys equations (1) and (2) and the boundary condition in $x = 0$

$$T(r,0) = T_0^{(2)}(r). \tag{10}$$

Then, the following conditions hold, for every r:

$$\operatorname*{Lim}_{x\to+\infty} \frac{\partial}{\partial x}[T_1(r,x) - T_2(r,x)] = 0 \tag{11}$$

$$\operatorname*{Lim}_{x\to+\infty} [T_1(r,x) - T_2(r,x)] = A \tag{12}$$

where A is a constant.

In fact, the function $\tilde{T}(r,x) = T_1(r,x) - T_2(r,x)$ obeys equations (6) and (7) and, as a consequence of equations (9) and (10), fulfils the boundary condition in $x = 0$

$$\tilde{T}(r,0) = T_0^{(1)}(r) - T_0^{(2)}(r). \tag{13}$$

Equations (6), (7) and (13) show that $\tilde{T}(r,x)$ can be interpreted as the temperature field of a fluid without viscous dissipation and axial thermal heat conduction, which flows in a circular duct with adiabatic walls and has a non-uniform temperature profile in the inlet section, $T_0^{(1)}(r) - T_0^{(2)}(r)$. Obviously, for every temperature distribution in the inlet section, if the wall is adiabatic and no viscous heat generation is present, the temperature field in the fluid tends to become uniform in the sections of the duct sufficiently far from $x = 0$. In other words, $\tilde{T}(r,x)$ tends to become axially invariant and its limit for $x \to +\infty$ is a constant, in agreement with equations (11) and (12).

ASYMPTOTIC TEMPERATURE FIELD

In this section, the asymptotic behaviour of the temperature field is analysed. In particular, a sufficient condition for the existence of an asymptotic thermally developed region is determined.

Let us denote by T_w and T_b, respectively the wall temperature and the bulk temperature. Then, the forced convection problem described in the preceding section will be said to allow a thermally developed asymptotic region if the dimensionless temperature $\vartheta = (T_w - T)/(T_w - T_b)$ becomes asymptotically invariant, i.e. if there exists a continuous and bounded function $\vartheta_\infty(r)$ with a continuous and bounded first derivative such that, for every r,

$$\operatorname*{Lim}_{x\to+\infty} \frac{T_w(x) - T(r,x)}{T_w(x) - T_b(x)} = \operatorname*{Lim}_{x\to+\infty} \vartheta(r,x) = \vartheta_\infty(r). \tag{14}$$

This definition is in agreement with the usual definition of thermally developed region stated, for instance, by Shah and London [9]. In fact, it is easily proved that if $\vartheta(r,x)$ is asymptotically invariant, then also the Nusselt number

$$Nu(x) = \frac{2r_0 q_w(x)}{k[T_w(x) - T_b(x)]} \tag{15}$$

is asymptotically invariant.

The bulk value of any function $X(r,x)$ is defined by the relation

$$X_b(x) = \frac{2}{\bar{u}r_0^2}\int_0^{r_0} X(r,x)u(r)r\,\mathrm{d}r. \tag{16}$$

Let us consider separately the cases of asymptotically vanishing and of asymptotically non-vanishing wall heat flux.

First case: asymptotically vanishing wall heat flux

Let us first assume that

$$\operatorname*{Lim}_{x\to+\infty} q_w(x) = 0. \tag{17}$$

We will prove that, if equation (17) holds, then the problem allows a completely developed asymptotic region in which the Nusselt number is zero. Let us consider a solution of equations (1) and (2), with the form

$$T(r,x) = \frac{2\alpha}{k\bar{u}r_0}\int_0^x q_w(x')\,\mathrm{d}x' + \frac{8\alpha\mu\bar{u}}{kr_0}\left[\frac{x}{r_0} + F(r,x)\right]. \tag{18}$$

By substituting equation (18) in equation (1), one obtains

$$\frac{\partial}{\partial r}\left(r\frac{\partial F}{\partial r}\right)=\frac{u(r)r}{\alpha}\left[\frac{\partial F}{\partial x}+\frac{q_w(x)}{4\mu\bar{u}^2}\right]+\frac{u(r)r}{\alpha r_0}-\frac{r_0}{8\alpha\bar{u}}r\Phi(r) \tag{19}$$

or, by means of equations (4) and (5),

$$\frac{\partial}{\partial r}\left(r\frac{\partial F}{\partial r}\right)=\frac{2\bar{u}}{\alpha}\left\{r\left(1-\frac{r^2}{r_0^2}\right)\left[\frac{\partial F}{\partial x}+\frac{q_w(x)}{4\mu\bar{u}^2}\right]+\frac{r}{r_0}-2\frac{r^3}{r_0^3}\right\}. \tag{20}$$

The substitution of equation (18) in equation (2) yields

$$\left.\frac{\partial F}{\partial r}\right|_{r=r_0}=\frac{r_0}{8\alpha\mu\bar{u}}q_w(x). \tag{21}$$

The right-hand side of equation (21) is related to the local Brinkman number, defined as [10]

$$Br(x)=\frac{\mu\bar{u}^2}{2r_0q_w(x)}. \tag{22}$$

In fact, equation (21) can be rewritten in terms of the Brinkman number and of the Peclet number $Pe = 2\bar{u}r_0/\alpha$:

$$\left.\frac{\partial F}{\partial r}\right|_{r=r_0}=\frac{Pe}{32r_0Br(x)}. \tag{23}$$

Let us point out that, as a consequence of equation (22), equation (17) is equivalent to the condition that $Br(x)\to\pm\infty$ when $x\to+\infty$.

Let us assume that there exists a solution of equations (20) and (23) such that, for every r,

$$\operatorname*{Lim}_{x\to+\infty}\frac{\partial F(r,x)}{\partial x}=0 \tag{24}$$

$$\operatorname*{Lim}_{x\to+\infty}F(r,x)=f(r) \tag{25}$$

where $f(r)$ is an analytic function of r. This assumption is legitimate. In fact, on account of equations (17), (24) and (25), in the limit $x\to+\infty$ equations (20) and (21) take the form

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}f}{\mathrm{d}r}\right)=\frac{2\bar{u}}{\alpha}\left(\frac{r}{r_0}-2\frac{r^3}{r_0^3}\right) \tag{26}$$

$$\left.\frac{\mathrm{d}f}{\mathrm{d}r}\right|_{r=r_0}=0. \tag{27}$$

Equations (26) and (27) have infinite solutions, but the difference between any pair of solutions is a constant. A solution of equations (26) and (27) is given by

$$f(r)=-\frac{Pe}{16}\left(2\frac{r^4}{r_0^4}-4\frac{r^2}{r_0^2}+1\right). \tag{28}$$

In this solution, the additive constant has been chosen so that the bulk value of $f(r)$ is zero. By employing equations (18), (25) and (28), one can thus conclude that the asymptotic expression of the temperature field has the form

$$T(r,x)=A+\frac{2\alpha}{k\bar{u}r_0}\int_0^x q_w(x')\,\mathrm{d}x' + \frac{8\alpha\mu\bar{u}}{kr_0}\left[\frac{x}{r_0}-\frac{Pe}{16}\left(2\frac{r^4}{r_0^4}-4\frac{r^2}{r_0^2}+1\right)\right] \tag{29}$$

where A is a constant determined by the boundary condition at $x = 0$. Since the bulk value of $f(r)$ is zero, from equations (28) and (29) one can easily deduce that the bulk temperature is given by

$$T_b(x)=A+\frac{2\alpha}{k\bar{u}r_0}\int_0^x q_w(x')\,\mathrm{d}x'+\frac{8\alpha\mu\bar{u}}{kr_0^2}x. \tag{30}$$

Equation (30) shows that $A = T_b(0)$, i.e. that the additive constant A represents the bulk value of the temperature profile prescribed in $x = 0$. By employing equation (29), it is easily verified that condition (14) is fulfilled and that the function $\vartheta_\infty(r)$ is given by

$$\vartheta_\infty(r)=2\frac{r^4}{r_0^4}-4\frac{r^2}{r_0^2}+2. \tag{31}$$

This dimensionless temperature profile is represented in Fig. 1.

From equation (31) and from the well known relation between Nu and $\vartheta(r)$ reported, for instance, by Bejan [11], one can obtain that the asymptotic value of the Nusselt number is zero, i.e.

$$\operatorname*{Lim}_{x\to+\infty}Nu=-2r_0\left.\frac{\mathrm{d}\vartheta_\infty}{\mathrm{d}r}\right|_{r=r_0}=0. \tag{32}$$

The asymptotic distribution of dimensionless temperature given by equation (31) and the asymptotic value of Nu given by equation (32) hold for all the cases in which the wall heat flux tends to zero when $x\to+\infty$.

Second case: asymptotically non-vanishing wall heat flux

Let us now assume that

$$\operatorname*{Lim}_{x\to+\infty}q_w(x)\neq 0 \tag{33}$$

and that there exists a constant β such that $q_w(x)$ obeys the condition

$$\operatorname*{Lim}_{x\to+\infty}\frac{1}{q_w(x)}\frac{\mathrm{d}q_w(x)}{\mathrm{d}x}=\beta. \tag{34}$$

As a consequence of equation (33), there exists a real number Br_∞ such that

$$\operatorname*{Lim}_{x\to+\infty}Br(x)=\operatorname*{Lim}_{x\to+\infty}\frac{\mu\bar{u}^2}{2r_0q_w(x)}=Br_\infty. \tag{35}$$

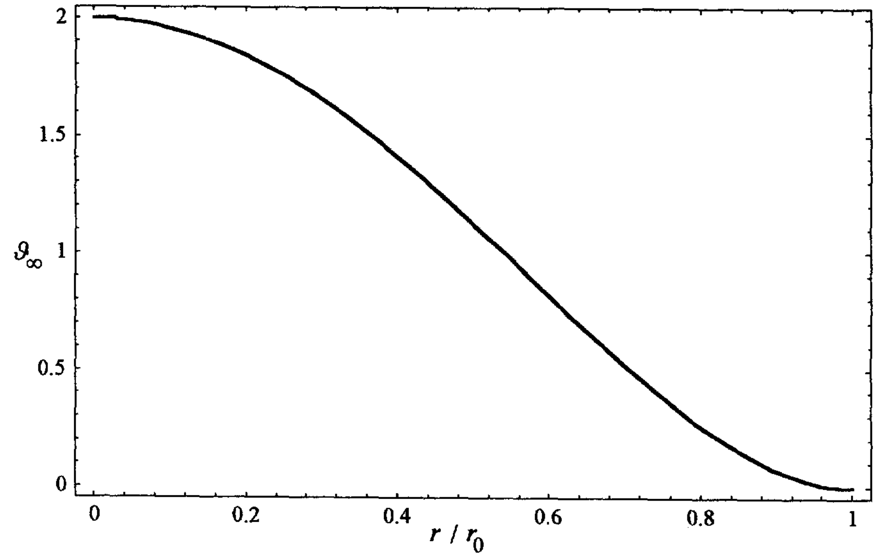

Fig. 1. Function ϑ_∞ vs r/r_0, for asymptotically vanishing wall heat flux.

It is important to note that the parameters β and Br_∞ are not mutually independent. Indeed, in Appendix it is proved that if $Br_\infty \neq 0$ and if equation (34) holds, then $\beta = 0$. Therefore, if equation (34) holds with $\beta \neq 0$, then $Br_\infty = 0$ (i.e. the limit of $q_w(x)$ for $x \to +\infty$ must be $\pm\infty$). In Appendix it is also proved that if equations (33) and (34) hold, then $\beta \geqslant 0$. Let us consider a solution of equations (1) and (2) with the form

$$T(r,x) = \frac{\mu \bar{u}^2}{6k}\left(1 - 6\frac{r^4}{r_0^4}\right) + \frac{2\alpha}{k\bar{u}r_0}\int_0^x q_w(x')\,\mathrm{d}x' + \frac{8\alpha\mu\bar{u}}{kr_0}\left[\frac{x}{r_0} + \frac{F(r,x)}{Br(x)}\right]. \quad (36)$$

By substituting equation (36) in equation (1) and by employing equations (4) and (5), one obtains

$$\frac{\partial}{\partial r}\left(r\frac{\partial F}{\partial r}\right) = Pe\left(\frac{r}{r_0} - \frac{r^3}{r_0^3}\right)\left[\frac{\partial F}{\partial x} + \frac{1}{q_w(x)}\frac{\mathrm{d}q_w(x)}{\mathrm{d}x}F + \frac{1}{8r_0} + \frac{Br(x)}{r_0}\right]. \quad (37)$$

The substitution of equation (36) in equation (2) yields

$$\left.\frac{\partial F}{\partial r}\right|_{r=r_0} = \frac{Pe}{32r_0}[1 + 8Br(x)]. \quad (38)$$

Let us assume that there exists a solution $F(r,x)$ of equations (37) and (38) which fulfils conditions (24) and (25) for every r, and such that $f(r)$ is an analytic function of r. This assumption is legitimate. In fact, on account of equations (24), (25), (34) and (35), the limit for $x \to +\infty$ of equations (37) and (38) is given by

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}f}{\mathrm{d}r}\right) = Pe\left(\frac{r}{r_0} - \frac{r^3}{r_0^3}\right)\left[\beta f + \frac{1 + 8Br_\infty}{8r_0}\right] \quad (39)$$

$$\left.\frac{\mathrm{d}f}{\mathrm{d}r}\right|_{r=r_0} = \frac{Pe}{32r_0}[1 + 8Br_\infty]. \quad (40)$$

Let us first consider the case $\beta > 0$. It has been already pointed out that, in this case, $Br_\infty = 0$. Moreover, in ref. [1] it is proved that the general solution of equation (39), with $Br_\infty = 0$, which obeys the symmetry condition $\mathrm{d}f/\mathrm{d}r|_{r=0} = 0$ can be expressed by means of the power series

$$f(r) = a_0 \sum_{n=0}^{\infty} c_n \left(\frac{r}{r_0}\right)^{2n} - \frac{1}{8\beta r_0}. \quad (41)$$

The coefficients c_n which appear in equation (41) are defined by the recursion formula

$$c_0 = 1;\quad c_1 = \frac{\hat{\beta}}{4};\quad \ldots;\quad c_n = \frac{\hat{\beta}}{(2n)^2}(c_{n-1} - c_{n-2}),\quad n \geqslant 2 \quad (42)$$

where $\hat{\beta} = Per_0\beta$. By integrating both sides of equation (39) with respect to r in the interval $[0, r_0]$ and by employing equation (40) and the symmetry condition $\mathrm{d}f/\mathrm{d}r|_{r=0} = 0$, it is easily proved that for $Br_\infty = 0$ one obtains $f_b = 0$. In ref. [1], it is proved that the power series which appears at the right-hand side of equation (41) has an infinite radius of convergence. The constant a_0 is determined by the boundary condition (40) and can be expressed as

$$a_0 = \frac{Pe}{64}\left(\sum_{n=1}^{\infty} nc_n\right)^{-1}. \quad (43)$$

On the other hand, it is easily proved that, if $\beta = 0$, equations (39) and (40) have infinite solutions which fulfil the symmetry condition $df/dr|_{r=0} = 0$ and are such that the difference between any pair of solutions is a constant. Let us choose the solution

$$f(r) = \frac{Pe}{128}(1+8Br_\infty)\left(4\frac{r^2}{r_0^2} - \frac{r^4}{r_0^4} - \frac{7}{6}\right) \quad (44)$$

which has a vanishing bulk value.

By employing equations (25) and (36), it is possible to conclude that the asymptotic expression of the temperature field has the form

$$T(r,x) = A + \frac{\mu \bar{u}^2}{6k}\left(1 - 6\frac{r^4}{r_0^4}\right) + \frac{2\alpha}{k\bar{u}r_0}\int_0^x q_w(x')\,dx' + \frac{8\alpha\mu\bar{u}}{kr_0}\left[\frac{x}{r_0} + \frac{f(r)}{Br(x)}\right] \quad (45)$$

where $f(r)$ is given by equation (41) for $\beta > 0$, or by equation (44) for $\beta = 0$, while A is a constant determined by the boundary condition in $x = 0$.

Since the bulk value of $f(r)$ is zero, from equation (45) it is easily proved that the bulk temperature is given by

$$T_b(x) = A + \frac{2\alpha}{k\bar{u}r_0}\int_0^x q_w(x')\,dx' + \frac{8\alpha\mu\bar{u}}{kr_0^2}x. \quad (46)$$

On account of equation (46), also in this case $A = T_b(0)$. By employing equations (41), (43) and (45) it is easily verified that, if $\beta > 0$, condition (14) is fulfilled and the function $\vartheta_\infty(r)$ can be expressed as

$$\vartheta_\infty(r) = \frac{f(r_0) - f(r)}{f(r_0)} = \left[\sum_{n=0}^{\infty} c_n\left(1 - \frac{8n}{\hat{\beta}}\right)\right]^{-1} \sum_{n=1}^{\infty} c_n\left[1 - \left(\frac{r}{r_0}\right)^{2n}\right]. \quad (47)$$

Representation of $\vartheta_\infty(r)$ vs r/r_0 for $\hat{\beta} = 10$, $\hat{\beta} = 100$ and $\hat{\beta} = 1000$ are reported in Fig. 2.

On the other hand, if $\beta = 0$, from equation (44) one obtains the following expression of $\vartheta_\infty(r)$:

$$\vartheta_\infty(r) = \frac{6}{11+48Br_\infty}\left[(1+16Br_\infty)\frac{r^4}{r_0^4} - 4(1+8Br_\infty)\frac{r^2}{r_0^2} + 3 + 16Br_\infty\right]. \quad (48)$$

Representations of $\vartheta_\infty(r)$ vs r/r_0 for $\beta = 0$ and for some values of Br_∞ are reported in Fig. 3. As it can be proved by means of equation (48), the curves reported in Fig. 3 cross each other at $r/r_0 = 1/2$.

Equation (47) defines a continuous and bounded function of r with a continuous and bounded first derivative, in the interval $0 \leqslant r \leqslant r_0$, for every positive value of $\hat{\beta}$ such that $f(r_0) \neq 0$. The same regularity holds for the function of r defined by equation (48), if $Br_\infty \neq -11/48$. Therefore, equation (14) is fulfilled for $\beta = 0$ and $Br_\infty \neq -11/48$, as well as for every positive value of $\hat{\beta}$ such that $f(r_0) \neq 0$. It is possible to check numerically that $f(r_0) \neq 0$ for every value of $\hat{\beta}$ in the interval $0 < \hat{\beta} \leqslant 10^5$. On account of equations (47) and (48), two forced convection problems with different distributions of the wall heat flux which fulfil equations (33) and (34) yield the same asymptotic profile of dimensionless temperature $\vartheta_\infty(r)$ if they have the same values of $\hat{\beta} = Per_0\beta$ and of Br_∞. As a consequence, they yield also the same asymptotic value of the Nusselt number, which can be evaluated through the expression [11]

$$\lim_{x\to+\infty} Nu = -2r_0 \left.\frac{d\vartheta_\infty}{dr}\right|_{r=r_0}. \quad (49)$$

Equations (47) and (49) yield

$$\lim_{x\to+\infty} Nu = 4\left[\sum_{n=0}^{\infty} c_n\left(1 - \frac{8n}{\hat{\beta}}\right)\right]^{-1} \sum_{n=1}^{\infty} nc_n \quad (50)$$

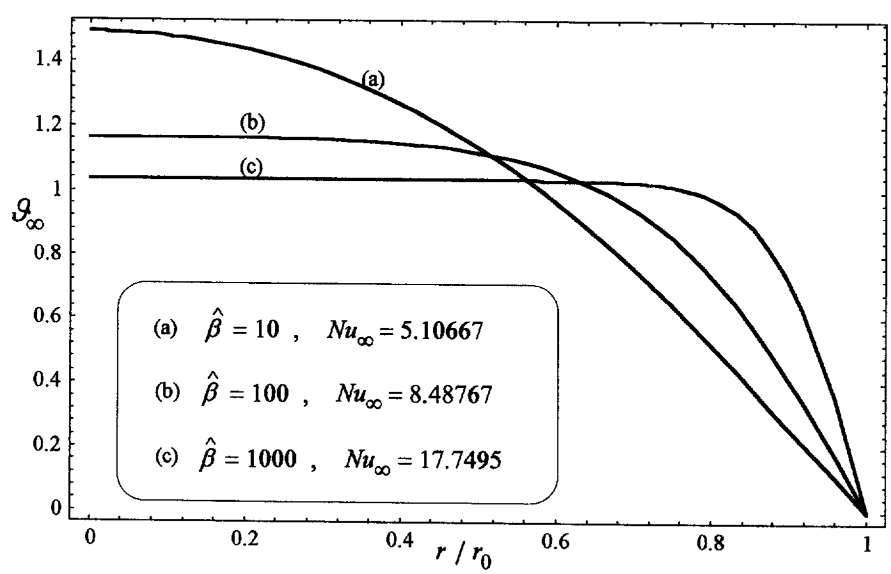

Fig. 2. Function ϑ_∞ vs r/r_0, for $\hat{\beta} > 0$.

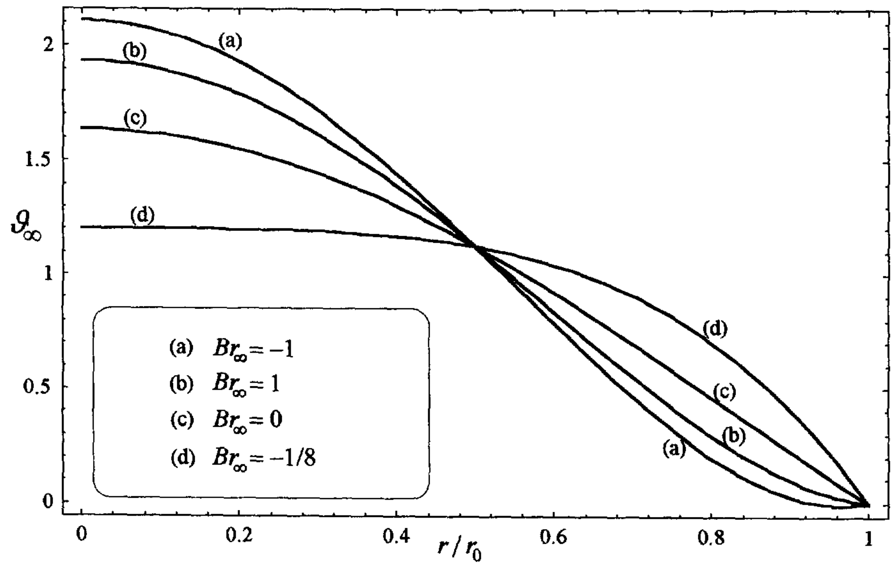

Fig. 3. Function ϑ_∞ vs r/r_0, for $\hat{\beta} = 0$.

when $\beta > 0$. On the other hand, when $\beta = 0$ equations (48) and (49) yield

$$\lim_{x \to +\infty} Nu = \frac{48}{11 + 48 Br_\infty}. \quad (51)$$

Values of the asymptotic Nusselt number, Nu_∞, in correspondence of $\hat{\beta} = 10$, $\hat{\beta} = 100$ and $\hat{\beta} = 1000$ are reported in Fig. 2.

Obviously, equation (50) refers to a case, $Br_\infty = 0$, in which viscous dissipation does not influence the asymptotic behaviour of the dimensionless temperature field. Indeed, the expression of the asymptotic value of Nu which appears in equation (50) coincides with that obtained by Barletta and Zanchini [1] in the absence of viscous dissipation, and is equivalent to that employed by Roetzel [12] in the analysis of laminar forced convection with an exponentially varying wall heat flux. Moreover, it has been proved in the literature [3, 4, 10] that $48/(11+48Br_\infty)$ represents the value of Nu_∞ in the case of uniform wall heat flux. In this paper, equations (50) and (51) assume a broader meaning. In fact, equations (33) and (34) are fulfilled with $\beta = 0$ not only when the wall heat flux is uniform, but also when $q_w(x)$ is given by: polynomial functions, rational functions in which the degree of the numerator is greater than or equal to the degree of the denominator, logarithmic functions, etc. Therefore, these distributions of the wall heat flux yield the existence of a completely developed asymptotic region in which the dimensionless temperature profile is given by equation (48) and the Nusselt number has the value $48/(11+48Br_\infty)$. In a similar way, equations (33) and (34) are fulfilled for $\beta > 0$ not only when the wall heat flux increases exponentially along the duct, but also when $q_w(x)$ is given by: the product of an exponential function and a polynomial function, the product of an exponential function and a rational function in which the degree of the numerator is greater than or equal to the degree of the denominator, etc. Therefore, for each of these distributions of wall heat flux there exists a fully developed asymptotic region in which the dimensionless temperature profile is given by equation (47) and the Nusselt number can be evaluated through equation (50), independently of the value of the fluid viscosity.

THE BOUNDARY CONDITIONS OF UNIFORM WALL TEMPERATURE AND OF CONVECTION WITH AN EXTERNAL FLUID

In this section, it is shown that the method developed in this paper allows a simple deduction of the results already available in the literature for the asymptotic value of the Nusselt number in the following boundary conditions:

(a) uniform wall temperature and
(b) external convection with a fluid having a uniform temperature, T_e, outside the boundary layer and a uniform convection coefficient h_e.

Therefore, this section provides a validation of the method proposed in this paper.

In order to determine the asymptotic behaviour of the temperature field and of the Nusselt number in the boundary condition (a), one can first evaluate the asymptotic behaviour of the wall heat flux. Thus, the problem is reduced to one of the cases studied in the preceding section. The method employed here is similar to that employed by Sparrow and Patankar in the absence of viscous dissipation [13].

By integrating both sides of equation (1) with respect to r in the interval $[0, r_0]$, one obtains the energy balance

$$\frac{\mathrm{d}T_b(x)}{\mathrm{d}x} = \frac{2\alpha}{k\bar{u}r_0} q_w(x) + \frac{8\alpha\mu\bar{u}}{kr_0^2}. \quad (52)$$

Since T_w = constant, equations (15) and (52) yield

$$\frac{\mathrm{d}}{\mathrm{d}x}(T_w - T_b) = -\frac{\alpha Nu}{\bar{u}r_0^2}(T_w - T_b) - \frac{8\alpha\mu\bar{u}}{kr_0^2}. \quad (53)$$

In the fully developed asymptotic region, $Nu = Nu_\infty$ is invariant along x, so that equation (53) can be easily integrated and yields

$$T_w - T_b(x) = C\exp\left(-\frac{\alpha Nu_\infty}{\bar{u}r_0^2}x\right) - \frac{8\mu\bar{u}^2}{kNu_\infty} \quad (54)$$

where C is an integration constant. By substituting equation (54) in equation (15) one has

$$q_w(x) = \frac{kNu_\infty}{2r_0}C\exp\left(-\frac{\alpha Nu_\infty}{\bar{u}r_0^2}x\right) - \frac{4\mu\bar{u}^2}{r_0}. \quad (55)$$

It is easily verified that $q_w(x)$ given by equation (55) fulfils conditions (33) and (34) with $\beta = 0$. Moreover, equations (35) and (55) yield $Br_\infty = -1/8$. Therefore, the asymptotic distribution of dimensionless temperature and the asymptotic value of Nu are given by equations (48) and (51) with $Br_\infty = -1/8$, i.e.

$$\vartheta_\infty(r) = \frac{6}{5}\left(1 - \frac{r^4}{r_0^4}\right) \quad (56)$$

$$\lim_{x\to-\infty} Nu = \frac{48}{5} = 9.6. \quad (57)$$

Equation (57) agrees with the value of the asymptotic Nusselt number obtained by Basu and Roy [3]. A representation of $\vartheta_\infty(r)$ in the case under exam can be found in Fig. 3, in correspondence of $Br_\infty = -1/8$. By employing equations (54), (56) and (57), one obtains easily

$$\lim_{x\to+\infty} T_b(x) = T_w + \frac{5\mu\bar{u}^2}{6k} \quad (58)$$

$$\lim_{x\to+\infty} T(r,x) = T_w + \frac{\mu\bar{u}^2}{k}\left(1 - \frac{r^4}{r_0^4}\right). \quad (59)$$

For the boundary condition (b), the wall heat flux can be expressed through the Biot number, $Bi = h_e r_0/k$, as follows:

$$q_w(x) = \frac{kBi}{r_0}(T_e - T_w). \quad (60)$$

As a consequence of equations (15) and (60), $q_w(x)$ can also be expressed as

$$q_w(x) = \frac{kBiNu}{r_0(2Bi+Nu)}(T_e - T_b). \quad (61)$$

By substituting equation (61) in equation (52), one obtains

$$\frac{\mathrm{d}}{\mathrm{d}x}(T_e - T_b) = -\frac{2\alpha BiNu}{\bar{u}r_0^2(2Bi+Nu)}(T_e - T_b) - \frac{8\alpha\mu\bar{u}}{kr_0^2}. \quad (62)$$

In the asymptotic thermally developed region, $Nu = Nu_\infty$ is invariant along x, so that equation (62) can be easily integrated and yields

$$T_e - T_b(x) = C\exp\left(-\frac{2\alpha BiNu_\infty}{\bar{u}r_0^2(2Bi+Nu_\infty)}x\right) - \frac{4\mu\bar{u}^2(2Bi+Nu_\infty)}{kBiNu_\infty} \quad (63)$$

where C is an integration constant. By substituting equation (63) in equation (61) one has

$$q_w(x) = \frac{kBiNu_\infty}{r_0(2Bi+Nu_\infty)}C\exp\left(-\frac{2\alpha BiNu_\infty}{\bar{u}r_0^2(2Bi+Nu_\infty)}x\right) - \frac{4\mu\bar{u}^2}{r_0}. \quad (64)$$

It is easily proved that $q_w(x)$ given by equation (64) fulfils conditions (33) and (34) with $\beta = 0$. Moreover, by employing equations (35) and (64) one obtains $Br_\infty = -1/8$. Therefore, equations (56) and (57) still hold, independently of the value of Bi, in agreement with the results obtained by Lin *et al.* [7]. As a consequence of equations (56), (57), (60) and (61), one obtains

$$\lim_{x\to+\infty} T_w(x) = T_e + \frac{4\mu\bar{u}^2}{k\,Bi} \quad (65)$$

$$\lim_{x\to+\infty} T_b(x) = T_e + \left(\frac{5}{6} + \frac{4}{Bi}\right)\frac{\mu\bar{u}^2}{k} \quad (66)$$

$$\lim_{x\to+\infty} T(r,x) = T_e + \frac{4\mu\bar{u}^2}{kBi} + \frac{\mu\bar{u}^2}{k}\left(1 - \frac{r^4}{r_0^4}\right). \quad (67)$$

CONCLUSIONS

The effect of viscous dissipation on the asymptotic behaviour of laminar forced convection in circular tubes has been analysed, under the assumption that the axial conduction in the fluid can be neglected. The results, schematically reported in Table 1, can be summarized as follows. A fully developed asymptotic region exists if, for $x \to +\infty$, one of the following conditions holds: (a) the wall heat flux $q_w(x)$ tends to zero, i.e. the local Brinkman number $Br(x)$ tends to infinity; (b) the wall heat flux does not tend to zero, and the quantity $(1/q_w)(\mathrm{d}q_w/\mathrm{d}x)$ tends to a finite limit β. If condition (a) is fulfilled, then the asymptotic value of the Nusselt number is zero. If condition (b) is fulfilled, then $\beta \geqslant 0$ and the following results are possible: if $\beta > 0$, $Br(x)$ tends to zero and the asymptotic value of the Nusselt number is not influenced by viscous dissipation effects; if $\beta = 0$ the asymptotic value of the Nusselt number depends on the asymp-

Table 1. Asymptotic values of the Nusselt number and asymptotic dimensionless temperature profiles, for different boundary conditions

Boundary condition	Br_∞	$\hat{\beta}$	Nu_∞	ϑ_∞
	$\pm\infty$	—	0	$2\frac{r^4}{r_0^4}-4\frac{r^2}{r_0^2}+2$
Prescribed wall heat flux	0	>0	$\frac{4\sum_{n=1}^{\infty} nc_n}{\sum_{n=0}^{\infty} c_n\left(1-\frac{8n}{\hat{\beta}}\right)}$	$\frac{\sum_{n=1}^{\infty} c_n\left[1-\left(\frac{r}{r_0}\right)^{2n}\right]}{\sum_{n=0}^{\infty} c_n\left(1-\frac{8n}{\hat{\beta}}\right)}$
	Any real value	0	$\frac{48}{11+48Br_\infty}$	$\frac{6}{11+48Br_\infty}\left[(1+16Br_\infty)\frac{r^4}{r_0^4}-4(1+8Br_\infty)\frac{r^2}{r_0^2}+3+16Br_\infty\right]$
Uniform wall temperature	$-1/8$	0	48/5	$\frac{6}{5}\left(1-\frac{r^4}{r_0^4}\right)$
External convection	$-1/8$	0	48/5	$\frac{6}{5}\left(1-\frac{r^4}{r_0^4}\right)$

totic value Br_∞ of the Brinkman number. In particular, the boundary conditions of uniform wall temperature and of heat transfer by convection to an external fluid fulfil condition (b) with $\hat{\beta}=0$ and $Br_\infty=-1/8$, and yield the same asymptotic value of the Nusselt number, namely $Nu_\infty=48/5$. Obviously, in these boundary conditions, as well as if condition (a) is fulfilled, it is completely wrong to neglect the effect of viscous dissipation on the asymptotic behaviour of the forced convection problem. For each case in which an asymptotic fully developed region exists, the analytic expression of the asymptotic temperature profile is reported in Table 1.

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APPENDIX

Let us first prove that, if there exist two real constants, $q_\infty \neq 0$ and β, such that

$$\lim_{x\to+\infty} q_w(x) = q_\infty \tag{A1}$$

and

$$\lim_{x\to+\infty} \frac{1}{q_w(x)}\frac{dq_w(x)}{dx} = \beta \tag{A2}$$

then $\beta = 0$.

On account of equation (A1), equation (A2) can be rewritten as

$$\lim_{x\to+\infty} \frac{\mathrm{d}q_w(x)}{\mathrm{d}x} = \beta q_\infty = a \in \Re. \quad (A3)$$

Let us consider the sequence $x_n = n$, which is such that

$$\lim_{n\to+\infty} x_n = +\infty. \quad (A4)$$

As a consequence of equation (A1),

$$\lim_{n\to+\infty} q_w(x_n) = q_\infty. \quad (A5)$$

On account of the theorem of the mean [14], $\forall n \exists c_n \in [n, n+1]$ such that

$$\frac{\mathrm{d}q_w}{\mathrm{d}x}(c_n) = \frac{q_w(x_{n+1}) - q_w(x_n)}{x_{n+1} - x_n} = q_w(x_{n+1}) - q_w(x_n). \quad (A6)$$

Equations (A5) and (A6) yield

$$\lim_{n\to+\infty} \frac{\mathrm{d}q_w}{\mathrm{d}x}(c_n) = 0. \quad (A7)$$

Since the limit of $\mathrm{d}q_w(x)/\mathrm{d}x$ exists, on account of equation (A3), equation (A7) proves the thesis.

Let us now prove that, if equation (A2) holds with $\beta < 0$ and if there exists the limit for $x \to +\infty$ of $q_w(x)$, then

$$\lim_{x\to+\infty} q_w(x) = 0. \quad (A8)$$

If the limit for $x \to +\infty$ of $q_w(x)$ exists, there are four possibilities: the limit is a non-vanishing constant, the limit is $+\infty$, the limit is $-\infty$, the limit is zero. The first possibility can be excluded, because we have proved above that, in this case, $\beta = 0$. The second possibility can be excluded because, for sufficiently high values of x, $q_w(x)$ would be positive and increasing. This circumstance is in contrast with the hypothesis that the limit which appears in equation (A2) is negative. The third possibility can be excluded by a similar argument. In fact, for sufficiently high values of x, one would have $q_w(x)$ negative and decreasing, in contrast with the hypothesis that the limit which appears in equations (A2) is negative. Therefore, the fourth possibility holds.